Review : Optimal detector

Goal : convey "W" ← the message or the message index.

There are M possible values for W : 1, 2, ..., M ← M-ary scheme

$W = i$ occurs with probability $p_i = P[W = i]$

To transmit $W = i$, the digital modulator send $S(t) = s_i(t)$ into the physical waveform channel:

$R(t) = S(t) + N(t)$

Because W is random, the input waveform S(t) is also random. It inhirits the same prior probabilities from W:

$$P[S(t) = s_i(t)] = p_i .$$

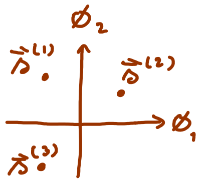
To simplify analysis, we convert the waveform channel into vector channel

$$\vec{R} = \vec{S} + \vec{N}$$

Again, there are M possible $\vec{S}$ vectors.

To find what they are, use GSOP on the signal set $\{s_1(t), s_2(t), \dots s_M(t)\}$ to get the orthonormal basis functions $\phi_1(t), \phi_2(t), \dots, \phi_K(t)$.

This enables us to write $s_i(t)$ as K-D vector $\vec{s}^{(i)}$.



We can then draw the M vectors $\vec{s}^{(1)}, \dots, \vec{s}^{(M)}$ as points in K-D space to visualize their arrangement. This is called the corresponding constellation for the digital modulator (scheme).

For the noise, if N(t) is an additive white Gaussian noise (AWGN) process with $PSD \equiv \frac{N_0}{2}$,

then the components of $\vec{N}$ are i.i.d. $\mathcal{N}(0, \sigma^2)$ where $\sigma^2 = \frac{N_0}{2}$:

$$f_{\vec{N}}(\vec{n}) = \frac{1}{(2\pi)^{K/2}\sigma^K} e^{-\frac{1}{2\sigma^2}\|\vec{n}\|^2} \leftarrow \vec{N} \sim \mathcal{N}(0, \sigma^2 I)$$

Note that there can be different models for noise.

Optimal Detector : At the receiver, we use a detector to detect W. The result may or may not be the same as W; so we call it $\hat{w}$.

The goal is to minimize $P(\mathcal{E}) = P[\hat{W} \neq W]$.

The detector that minimizes $P(\mathcal{E})$ is the MAP detector.

In this class, we assume independent additive noise. In which case,

$$\hat{w}_{MAP}(\vec{r}) = \arg\max_{i=1,\dots,M} \; p_i \, f_{\vec{N}}(\vec{r} - \vec{s}^{(i)}).$$

If p_i's are not used, then we have the ML detector

$$\hat{w}_{ML}(\vec{r}) = \arg\max_{i=1,\dots,M} \; f_{\vec{N}}(\vec{r} - \vec{s}^{(i)}).$$

Remark: When the messages are equiprobable, $\hat{w}_{MAP} = \hat{w}_{ML}$.

Suppose $\vec{N} \sim \mathcal{N}(0, \sigma^2 I)$, then

$$\hat{w}_{MAP}(\vec{r}) = \arg\min_{i=1,\dots,M} \; \|\vec{r} - \vec{s}^{(i)}\|^2 - 2\sigma^2 \ln p_i$$

$$= \arg\max_{i=1,\dots,M} \; \langle \vec{r}, \vec{s}^{(i)} \rangle + \sigma^2 \ln p_i - \tfrac{1}{2} E_i$$

$= \langle r(t), s_i(t) \rangle$

can be implemented via matched filter

$E_i = \|\vec{s}^{(i)}\|^2 = \|s_i(t)\|^2$

When the messages are equiprobable

or

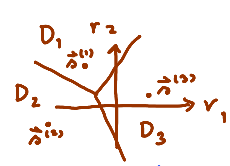
when the ML detector is used,

$$\hat{w}(\vec{r}) = \arg\min_{i=1,\dots,M} \; \|\vec{r} - \vec{s}^{(i)}\|$$ = minimum-distance detector

$= d(\vec{r}, \vec{s}^{(i)})$

This is the detector that we use in chapter 11 to derive the Q matrix.

Decision Regions $\quad D_i = \{\vec{r} : \hat{w}(\vec{r}) = i\}$



When minimum-distance detector is used, the boundaries of the decision regions can be found from the perpendicular bisector of the line connecting each pair of two signal points.

$$P(\mathcal{E}) = P[\hat{W} \neq W] = \sum_{i=1}^{M} \underbrace{P[\hat{W} \neq i \mid W = i]}_{P(\mathcal{E}|i)} p_i = 1 - \sum_{i=1}^{M} \underbrace{P[\hat{W} = i \mid W = i]} p_i$$

In chapter 11, this is the (i,i)-element in the Q matrix.

Review : norm, inner-product, energy

$x(t)$: $E_x = \|x(t)\|^2 = \langle x(t), x(t) \rangle = \int_{-\infty}^{\infty} |x(t)|^2 dt$

$\langle x_1(t), x_2(t) \rangle = \int_{-\infty}^{\infty} x_1(t)\, x_2^*(t)\, dt$

$\vec{x}$: $E_x = \|\vec{x}\|^2 = \langle \vec{x}, \vec{x} \rangle = \vec{x}^H \vec{x}$

$= (x_1^* \cdots x_K^*) \begin{pmatrix} x_1 \\ \vdots \\ x_K \end{pmatrix} = \underbrace{x_1^* x_1}_{|x_1|^2} + x_2^* x_2 + \cdots + x_K^* x_K = \sum_{i=1}^{K} |x_i|^2$

$\langle \vec{x}_1, \vec{x}_2 \rangle = \vec{x}_2^H \vec{x}_1$

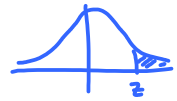
Monday, October 14, 2013 11:17 AM

Review : Some symbols that have several meanings

(depending on the way it is used)

Q

$Q(z) \leftarrow$ the Q function



$Q(y|x)$

Q matrix

$$
\left.\begin{array}{lll}
P_X(x) \longrightarrow & p(x) \longrightarrow & \underline{p} \\
P_{Y|X}(y|x) \longrightarrow & Q(y|x) \longrightarrow & Q \\
P_Y(y) \longrightarrow & q(y) \longrightarrow & \underline{q}
\end{array}\right\}
\qquad
\begin{array}{l}
q(y) = \sum_x p(x) Q(y|x) \\
\underline{q} = \underline{p} Q
\end{array}
$$

$$
Q = \begin{array}{c|cc}
x \backslash y & 1 & 2 \\
\hline
1 & 1/3 & 2/3 \\
2 & 4/5 & \frac{1}{5}
\end{array}
\qquad
Q(y|x) = \begin{cases}
1/3, & x=1, y=1, \\
2/3, & x=1, y=2, \\
4/5, & x=2, y=1, \\
1/5, & x=2, y=2, \\
0, & \text{otherwise}
\end{cases}
$$

H

$$H(X) = -\sum_x p_X(x) \log_2 p_X(x)$$

$$H(p) = -p \log_2 p - (1-p) \log_2 (1-p)$$

$$H(\underline{p}) = -\sum_i p_i \log_2 p_i$$

$H(X,Y)$

$H(Y|X)$

$H(X|Y)$

parity check matrix H

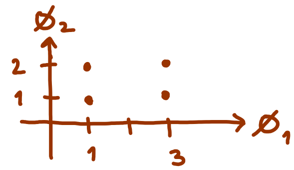
Review : Finding E_s.

Note that E_s = average energy per symbol

$$= \mathbb{E}\left[\|\vec{s}\|^2\right] = \mathbb{E}\left[\sum_{k=1}^{K} |s_k|^2\right] = \underbrace{\sum_{k=1}^{K} \mathbb{E}\left[|s_k|^2\right]}$$

So, we can find the average energy in each dimension and then add the results.

Example : Consider the constellation



In the first dimension, average energy is $\frac{1}{4}\left(1^2+1^2+3^2+3^2\right) = \frac{1^2+3^2}{2} = 5$

In the second dimension, average energy is $\frac{1}{4}\left(1^2+1^2+2^2+2^2\right) = \frac{1^2+2^2}{2} = \frac{5}{2} = 2.5$

Therefore, $E_s = 5 + 2.5 = 7.5$.

The average energy in 1-D when (1) there are M points,
(2) the points are equally spaced with adjacent distance $= d$, and
(3) the constellation is centered at 0,

is given by $E_s = \frac{d^2}{12}\left(M^2-1\right)$

see the derivation in HW3.

The derivation involves knowing the sum of the form $\sum_{k=1}^{M} k$ and $\sum_{k=1}^{M} k^2$.